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Nonlinear Programing by Projection-Restoration Applied to Optimal Geostationary Satellite Positioning

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This study is concerned with the development and application of a quasi-Newton descent algorithm for solving finite dimensional optimization problems subject to nonlinear equality and inequality constraints by the use of gradient projection and constraint restoration. The algorithm is applied to the positioning of a geostationary satellite which is interpreted as a problem of optimal N-impulse orbital rendezvous.

Introduction

THREE important considerations in the choice of a method for the numerical solution of a constrained optimization problem are accuracy, computing time, and ease of implementation. The gradient projection concept meets these requirements by providing a direct technique for extending gradient methods of unconstrained minimization. In contrast, penalty function methods attempt to solve a constrained problem by repeated unconstrained minimization and are often more costly to use. The gradient projection algorithm of Rosen[1] and a subsequent extension by Goldfarb[2] were specifically designed for use with linearly constrained problems. Nonlinear constraints were to be accommodated by the use of a restoration step.[3] Unfortunately, such a procedure is unsatisfactory if a minimum on the tangent plane approximation lies far from the constraint. This difficulty is overcome in a modification of the gradient projection concept as introduced by Kelley and Speyer[4,5] and Miele et al.[6,7] which penalizes large violations of the constraints along the search direction.

This paper is concerned with the use of quasi-Newton descent with the preceding modification for solving finite dimensional optimization problems subject to equality and inequality constraints. Development of the projection-restoration algorithm was motivated by a problem of positioning a geostationary satellite subjected to control errors. The positioning problem, excluding disturbances, is formulated as one of optimal N-impulse orbital rendezvous. By assuming control in terms of a specified number of impulsive velocity increments, a finite dimensional optimization problem is obtained. The stochastic optimization problem, including disturbances, is effectively solved by a gradient Monte Carlo search[8] that repeatedly utilizes the algorithm described in this paper. Finally, numerical results for the geostationary positioning problem and a well known Earth-Mars rendezvous example are presented. It is shown that the projection-restoration method using quasi-Newton descent is an effective technique for solving optimization problems arising in practical applications.

Problem Statement and Notation

Consider the nonlinear programing problem of computing an element x in X which minimizes a nonlinear functional $f(x)$ subject to equality constraints $g(x) = 0$ where g is a nonlinear mapping of X into Y. Assume that f and g are second continuously differentiable and that a relative minimum $\hat{x}$ exists. For the purposes of the application in this paper, X and Y are identified with the Euclidean spaces R^n and R^p, respectively, with the usual inner product $\langle\cdots\rangle_n$ and derived norm $\|\cdot\| = \langle\cdots\rangle_n^{1/2}$; for u and v in R^n

$$\langle u, v\rangle_n = \sum_{i=1}^{n} u_i v_i, \qquad \|u\|_n = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}$$

Relative to the standard basis, the element x may be represented as an n-tuple, the gradient f_x as an n-tuple, the Jacobian g_x as a $p \times n$ matrix whose rows are the gradients of the p elements of g and the adjoint $g_x{}^*$ as the matrix transpose of g_x.

Projection-Restoration

Assume that $p < n$ and that the p gradients of elements of g are linearly independent. Define the functional L by

$$L(x, \lambda) = f(x) + \langle g, \lambda\rangle_p \tag{1}$$

where λ, the Lagrange multiplier, is an element in R^p. A necessary condition that $L(x, \lambda)$ have a minimum is that its first differential

$$dL(x; \lambda; \delta x, \delta\lambda) = \langle L_x, \delta x\rangle_n + \langle L_\lambda, \delta\lambda\rangle_p$$

vanish at $x = \hat{x}$ and $\lambda = \hat{\lambda}$. This implies that $\hat{x}$ and $\hat{\lambda}$ satisfy the following equations: the constraint equation

$$g(\hat{x}) = 0 \tag{2}$$

and the minimization condition

$$f_x(\hat{x}) + g_x{}^*(\hat{x})\hat{\lambda} = 0 \tag{3}$$

One effective approach to the numerical solution of the original constrained minimization problem is a two-step procedure based on satisfying Eq. (2) while adjusting x by gradient descent to approximate a solution to Eq. (3). The procedure is repeated until a stopping condition is met.

Let a correction δx, called the restoration increment, to an initial guess x be chosen to satisfy Eq. (2) to first order

$$g(x) + g_x(x)\,\delta x = 0 \tag{4}$$

From Eq. (4), δx may be formally expressed as

$$\delta x = -g_x{}^+ g \tag{5}$$

where $g_x{}^+$ is a pseudo inverse of g_x and where, for brevity, the dependence of x has been dropped. Since $p < n$ there exists a

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solution to Eq. (4). In fact there are an infinite number of solutions each having the form $\delta y + \delta z$ where δy is a particular solution of Eq. (4) and δz is an element in the nullspace of g_x, i.e., satisfying $g_x \delta z = 0$. One way to ensure uniqueness is to use the pseudo inverse

$$g_x^{+} = g_x^*(g_x g_x^*)^{-1} \tag{6}$$

that selects the smallest value of δx, in the norm on R^n, that lies on the intersection of the tangent planes to the constraints. Linear independence of the gradients of elements of g ensures that $g_x g_x^*$ is invertible. Using Eq. (5), an improved value $\bar{x}$ is obtained from

$$\bar{x} = x - \alpha g_x^*(g_x g_x^*)^{-1} g \tag{7}$$

where α is a positive, scalar step length computed by linear search to minimize the constraint violation $\|g(x-\alpha g_x^{+} g)\|_p$.

A second correction $\delta\bar{x}$, the projection increment, is selected so that $L(\bar{x}, \lambda)$ is decreased in the direction of its negative gradient

$$\delta\bar{x} = -L_x(\bar{x}, \lambda) \tag{8}$$

The multiplier needed to compute Eq. (8) may be formally expressed from Eq. (3) in terms of a pseudo inverse $(g_x^*)^{+}$ as

$$\lambda = -(g_x^*)^{+} f_x \tag{9}$$

For $\bar{x} \neq \hat{x}$, there is no value of λ in R^p for which Eq. (3) is satisfied. An effective approximation, however, is obtained by use of the pseudo inverse

$$(g_x^*)^{+} = (g_x g_x^*)^{-1} g_x \tag{10}$$

that selects the least-squares value of λ in the sense of minimizing the residual

$$\|f_x + g_x^* \lambda\|_n^2 \tag{11}$$

Using Eqs. (8–10), $\bar{x}$ is updated according to the descent iteration

$$\begin{aligned} \bar{\bar{x}} &= \bar{x} - \beta(f_x + g_x^* \lambda) \\ &= \bar{x} - \beta P f_x \end{aligned} \tag{12}$$

where β is a positive, scalar step length computed by linear search to minimize $L(\bar{x} - \beta P f_x, \lambda)$, and

$$P = I - g_x^*(g_x g_x^*)^{-1} g_x \tag{13}$$

is an $n \times n$ matrix representing the operator that projects the gradient f_x into the manifold formed by the intersection of the constraint tangent planes. Repeated application of Eqs. (7) and (12) forms the basis of the projection-restoration algorithm (Fig. 1).

Nonorthogonal Projection

Terminal convergence of the steepest descent gradient search in Eq. (12) can be substantially improved by use of one of the well-known conjugate direction methods. A subclass of these methods is called quasi-Newton because, in addition to generating conjugate directions, a symmetric positive definite approximation H to the inverse Hessian L_{xx}^{-1}, if it exists, is obtained and used to compute an improved direction of descent.[9,10]

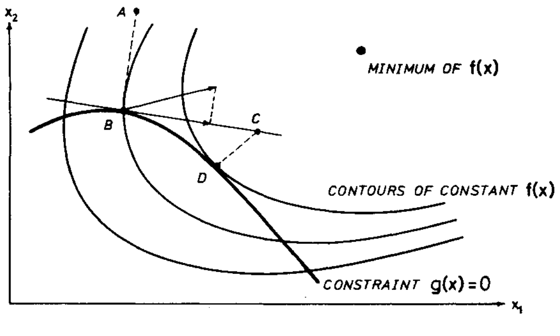

Fig. 1 Simplified example of projection-restoration method. A: Initial point. B: Constraint restoration. C: Minimum of linear search on tangent plane. D: Constraint restoration and constrained minimum.

The following modification of the gradient projection concept is equivalent to scaling the inner product so that a direction of descent computed from Eq. (12) points to an extremum of a second-order approximation to $L(x, \lambda)$. It was first described by Fletcher[11] and Kelley and Speyer.[4] For any symmetric positive definite $n \times n$ matrix G, define on X the nonorthogonal (weighted) inner product and derived norm, respectively,

$$\langle u, v\rangle_G = \langle u, Gv\rangle_n, \quad \|u\|_G = \langle u, Gu\rangle_n^{1/2}$$

It is easily verified that a given H, approximating L_{xx}^{-1}, generates the minimum norm pseudo inverse

$$g_x^{+} = H g_x^*(g_x H g_x^*)^{-1} \tag{14}$$

which minimizes $\|\delta x\|_{H^{-1}}$, and the least-squares pseudo inverse

$$(g_x^*)^{+} = (g_x H g_x^*)^{-1} g_x H \tag{15}$$

which minimizes $\|f_x + g_x^* \lambda\|_H^2$. Using Eqs. (14) and (15), the restoration and projection increments are, respectively,

$$\delta x = H g_x^*(g_x H g_x^*)^{-1} g \tag{16}$$

$$\begin{aligned} \delta\bar{x} &= H(f_x + g_x^* \lambda) \\ &= H P^H f_x \end{aligned} \tag{17}$$

where

$$P^H = I - g_x^*(g_x H g_x^*)^{-1} g_x H \tag{18}$$

is an $n \times n$ matrix representing a nonorthogonal projection operator. The inverse Hessian approximation H is updated after each successful projection update according to established formulas.[9] A good vector space interpretation of gradient projection relative to nonorthogonal inner products is given by Beltrami.[12]

Inequality Constraints

A more complete formulation of a nonlinear programing problem contains inequality constraints of the form $h(x) \leqq 0$ where h is a nonlinear mapping from X into Z. In the parameter optimization problem, Z is identified with the Euclidean space R^q. The simplest method for using gradient projection to solve problems having inequality constraints is to transform each inequality into an equality by use of slack variables. For example, a constraint on the first component of x, $x_1 \leqq 0$, is replaced by the equality $x_1 + s^2 = 0$, where s is a scalar and the minimization is to be performed in the enlarged space R^{n+1}. An obvious disadvantage of this method is that the dimension of the problem is increased and, accordingly, storage and computation time may become excessive. A second method for accommodating inequality constraints using gradient projection was formulated by Rosen[1] and has been recently extended to a class of nonlinear constraints, including conditions and rates of convergence, by Rosen and Kreuser.[13] A third possible method is to use only those gradients which correspond to linearly independent, active constraints at the current iteration.[14–16] At a constrained minimum, the Kuhn-Tucker conditions[9] imply that active constraints are those that are violated and correspond to non-negative Lagrange multipliers. Such formal modification of projection algorithms must be done with care, however. The active set must not change too often in order to avoid destroying the conjugate direction property of the quasi-Newton descent.

Dynamic Constraints

A typical class of problems in optimal control requires the determination of control parameters, related through known dynamical equations to the system state, which minimize the value of a cost functional involving control and state. If the number of control parameters is finite, there are two standard methods for evaluating the gradient of the cost functional.[17] The first method requires the solution of an initial value problem specified by the state sensitivity equations, the number of which increases linearly with the number of adjustable parameters. Numerical integration of these equations can be avoided in special cases for which an exact solution or close approximation can be evaluated, such as linear or two body dynamics. The second method requires solution of a two-point boundary value problem specified by the state and adjoint equations, the number of which is always twice the dimension of the state. If the number of control parameters is small, the first method usually involves less computation. For this reason, it was chosen for the application problem described in the following section.

More general problems of optimal control require formulation in an infinite dimensional space of control functions and solution of the adjoint system of equations (and the associated two-point boundary value problem) in order to compute the cost gradient. Evaluation of the projection matrix is complicated by the need to compute adjoints in Hilbert function space. The methods of steepest descent and conjugate gradients are easily extended to infinite dimensional spaces.[18] The corresponding extension of quasi-Newton descent, however, even for unconstrained minimization, is considerably more complicated.[19]

Optimal Positioning of a Geostationary Satellite

A problem of current interest concerns the positioning of a communications satellite at a specified point in a synchronous Earth orbit. The resulting trajectory rendezvous problem requires the determination of a minimum fuel control sequence and the corresponding optimal trajectory subject to 1) dynamic constraints of two body motion, 2) initial position and velocity on a specified transfer orbit, and 3) rendezvous with a point having specified position and velocity on a target orbit. To these constraints may be added further restrictions, such as bounds on the control magnitude, allowable control intervals, and fulfillment of visibility requirements from tracking stations.[20] Together, they form a set of nonlinear equality and inequality constraints. A similar formulation of a problem of orbital transfer has been given by Johnson[21] and Bean[22] using a method combining penalty functions and Newton iteration.

Define the state ‡

$$x(t) = (r(t), v(t)) \tag{19}$$

a six-tuple composed of the position and velocity at time $t \geqq t_o$. For given N, require that impulsive velocity increments c_i occur at times t_i, $i = 1, \ldots, N$. The resulting parameters

$$\pi = (t_1, c_1, \ldots, t_N, c_N) \tag{20}$$

then define an element in R^{4N} which is to be computed so that the total fuel consumption

$$f(\pi) = \sum_{i=1}^{N} \|c_i\| \tag{21}$$

is minimized subject to the following constraints.

Orbital Constraints

For the satellite positioning problem, the initial (transfer) orbit is a highly eccentric ellipse with apogee close to the synchronous radius. The initial state lies on this orbit and may be computed from the Kepler elements as

$$x_o(t_1) = h_o[a, e, i, \Omega, \omega, E(t_o), t_1] \tag{22}$$

where h_o is a known mapping of the orbital elements to rectangular coordinates in an Earth-centered, equatorial, inertial system.[23] The first switching time t_1 may be free or constrained. At or near apogee passage, the satellite is injected into a near synchronous (primary) orbit by a powerful apogee motor thrust c_1. By means of smaller corrective thrusts $c_2, \ldots, c_N$ the satellite is finally positioned at a specified target point with respect to the Earth. This procedure assumes exact orbit determination and error-free controls as well as impulsive approximation of the large apogee thrust. It was used to gain an understanding of the general problem and to determine the feasibility of solving it by the projection-restoration method. A more complete formulation including, for example, control errors, fixed spin-axis thrust directions, drift to the target point, and visibility requirements is currently being investigated.

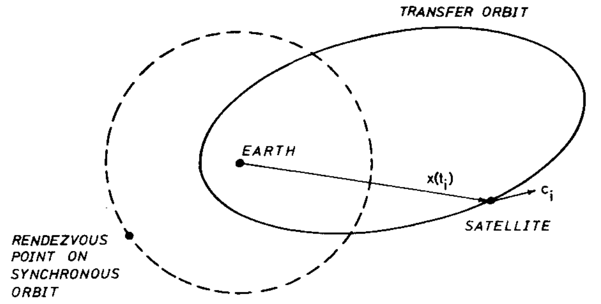

Fig. 2 Satellite positioning problem.

The final (target) orbit is a circle of given radius lying in the plane of the equator. The rendezvous condition is expressed by the equality constraint

$$g(x_N, t_N) = x_N(t_N) - h_f(t_N) = 0 \tag{23}$$

where $x_N(t)$ is the state after the Nth impulse and h_f is a known mapping of the orbital elements. In the case of circular motion, the elements Ω, ω, and $E(t)$ are not defined and are replaced by the longitude measured from a suitable reference line.

Dynamic Constraints

The preceding choice of control parameters allows great simplification in the formulation and solution of the dynamic constraints. The original N-impulse problem may now be represented equivalently as a sequence of initial value problems with the control parameters entering explicitly in the initial conditions

$$r_i(t_i) = r_{i-1}(t_i), \quad v_i(t_i) = v_{i-1}(t_i) + c_i \tag{24}$$

A generalized solution of the two-body equations has been developed by Goodyear.[24] This solution may be computed from the initial state $x_{i-1}(t_{i-1})$ and the time interval $t_i - t_{i-1}$. The important features of this solution are that all forms of two-body motion are included and only one generalized Kepler's equation must be solved. In order to evaluate the cost and constraint gradients, the transition matrix $\phi_i(t_i, t_{i-1})$ of first partial derivatives of the state is needed. This matrix can be computed in closed form after each velocity increment using the Goodyear method (Figs. 2 and 3).

Gradients

The gradients of the cost functional and constraints may be evaluated using the chain rule of differentiation in the forward direction of motion. The gradient matrices of the state with respect to the parameters are computed from the following relations:

$$\frac{\partial x_i(t_i)}{\partial \pi} = \frac{\partial x_i(t_i)}{\partial x_{i-1}(t_{i-1})} \frac{\partial x_{i-1}(t_{i-1})}{\partial \pi}$$
$$= \phi_i(t_i, t_{i-1}) \frac{\partial x_{i-1}(t_{i-1})}{\partial \pi}, \quad i = 2, \ldots, N \tag{25}$$

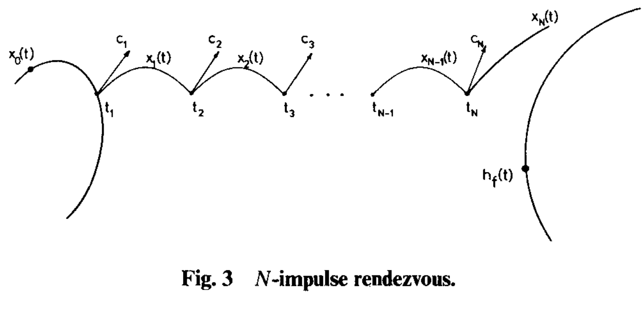

Fig. 3 N-impulse rendezvous.

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‡ In accordance with customary notation, $x(t)$ denotes the state and should not be confused with the parameter vector used in the derivation of the projection-restoration algorithm.

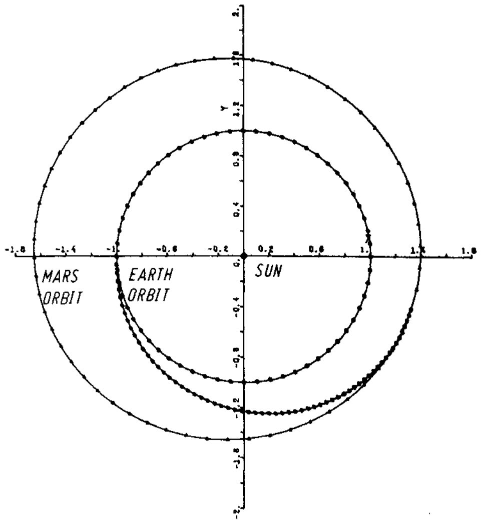

Fig. 4 Earth-Mars minimum fuel rendezvous.

where $\phi_i(t_i, t_{i-1})$ is the transition matrix for the $(i-1)$st arc. For the ith point of the trajectory, $i = 1, \ldots, N$

$$\frac{\partial x_i(t_i)}{\partial c_i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \frac{\partial x_i(t_i)}{\partial t_i} = (-c_i, 0, 0, 0) \tag{26}$$

For $j > i$

$$\partial x_i(t_i)/\partial c_j = \partial x_i(t_i)/\partial t_j = 0$$

For $j < i$, Eq. (25) yields

$$\frac{\partial x_i(t_i)}{\partial \pi_j} = \prod_{k=j+1}^{i} \phi_k(t_k, t_{k-1}) \frac{\partial x_j(t_j)}{\partial \pi_j} \tag{27}$$

where π_j denotes the parameter element (t_j, c_j).

Numerical Results

The method of projection-restoration using quasi-Newton descent together with the previous formulation was programed and applied to the following trajectory optimization problems. In all cases it is required to find minimum fuel rendezvous.

As a first test case, the now familiar Earth-Mars rendezvous example was chosen.[25] A trajectory consisting of an initial burn phase into interplanetary injection, followed by a coasting period, and finally by a second burn phase was specified. The resulting parameter optimization problem has dimension $n = 8$. In addition to the six rendezvous constraints, Eq. (23), the initial launch date was held fixed so that $p = 7$. The problem formulation of the last section may be applied without significant change. (In fact, the program is written to find optimal trajectories between arbitrary orbits.) The orbital data are shown in Table 1.

Table 1 Orbital data for Earth-Mars Problem

Initial orbit	Final orbit
$t_o = 0$ (12:00 noon May 9, 1971)	$a = 1.5236$ a.u.
$x_o(t_o) = \begin{pmatrix} -0.99980 \\ 0.02009 \\ 0.0 \\ 0.02009 \\ -0.99980 \\ 0.0 \end{pmatrix}$ a.u.	$e = 0.093393$ $i = 1.85002°$ $\Omega = 0°$ $\omega = 335.42°$ $E(t_o) = 243.56°$

Using a starting estimate of

$$t_1 = 0 \text{ days} \qquad c_1 = (-3.212, -3.705, -4.795) \text{ km/sec}$$
$$t_2 = 150 \qquad c_2 = (0, 6.858, 0.5488)$$

the optimal solution obtained was

$$\hat{t}_1 = 0 \qquad \hat{c}_1 = (-1.755, -2.191, 1.033)$$
$$\hat{t}_2 = 225.45 \qquad \hat{c}_2 = (1.693, 2.973, 0.096)$$

resulting in a total minimum fuel cost of 6.413 km/sec. A computer plot of the final converged trajectory is shown in Fig. 4.

For the satellite positioning problem, a series of typical cases arising in practice is presented. Rather than attempt an exhaustive analysis of the problem, here the main idea is to illustrate the ability of the projection-restoration method to solve a specific minimum fuel optimization example subject to various constraints. In all cases below, the orbital data shown in Table 2 apply.

Table 2 Orbital data for satellite positioning problem

Initial (transfer) orbit	Final orbit
$t_o = 0$ (apogee passage) $a = 25078$ km $e = 0.736047$ $i = 5°$ $\Omega = 180°$ $\omega = 180°$ $E(t_o) = 180°$	$h_f(t_o) = \begin{pmatrix} -42164.22 \\ 0.00 \\ 0.00 \\ 0.00 \\ -3074.65 \\ 0.00 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \text{km} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \text{m/sec} \end{matrix}$

The x axis points to the descending node of the transfer orbit in the equatorial plane (Fig. 5). All times are measured from apogee passage in the transfer orbit.

Problem 1 is used as a basis for comparison; a two-impulse ($n = 8$) minimum fuel transfer-target trajectory is calculated. The required apogee thrust is seen from Table 3 to be 1463.5 m/sec with the final correction $c_2 = 41.1$ m/sec occurring 13.08 hr after insertion into the primary orbit.

Problem 2 shows the savings in total fuel that may be realized by allowing an additional corrective impulse in the secondary orbit ($n = 12$). Since any such gains accrue to the satellite's stationkeeping capability, a savings of close to 20 m/sec indicates

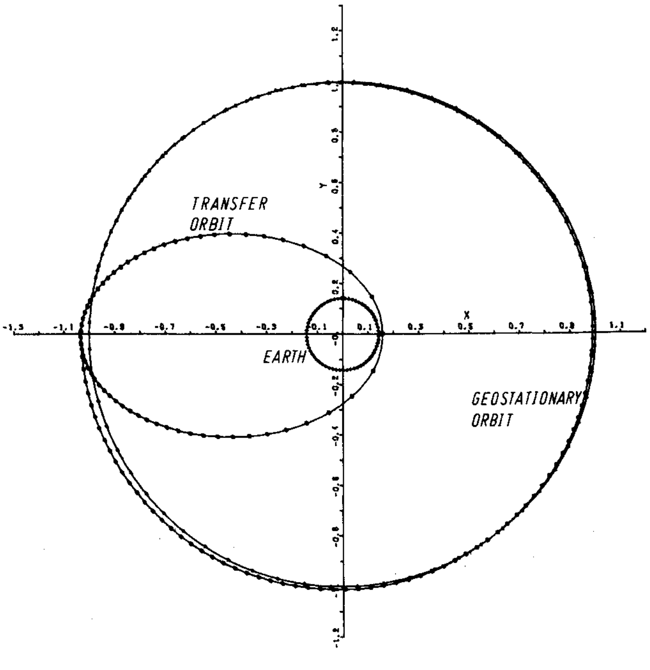

Fig. 5 Three-impulse satellite positioning.

Table 3 Satellite positioning (rendezvous)

	Initial values	Converged values Problem 1	Problem 2	Problem 3	Problem 4[a]	Problem 5[a]
t_1	0 hr	−0.242	−0.143	0.017	0.0	0.0
c_1	0 m/sec	125.5	72.3	1.94	0.0	0.0
	−1420	−1452.7	−1443.8	−1435.6	−1424.5	−1424.5
	0	−125.4	−128.9	−111.9	−124.6	−124.6
t_2	12 hr	12.84	11.68	11.33	9.54	8.0
c_2	0 m/sec	−26.4	−0.04	17.6	−0.76	14.4
	−50	−29.6	−25.6	−29.3	−13.8	−50.8
	0	10.6	4.58	16.7	16.2	14.9
t_3	24 hr		23.2	22.8	27.3	25.0
c_3	0 m/sec		−0.40	−1.06	−45.8	−51.7
	−20		−7.8	−14.5	−70.7	−30.2
	0		−2.26	−8.5	−2.46	−3.67
		1463.5	1451.4	1440.0	1430.0	1430.0
		41.1	25.9	38.1	21.3	54.8
			8.1	16.8	84.2	60.0
	Total cost m/sec	1504.6	1485.4	1494.9	1535.5	1544.8

[a] For these cases, the first parameter vector (t_1, c_1) is fixed (not optimized).

the advisability of designing for a three impulse mission. A computer plot of the final converged trajectory is shown in Fig. 5.

Problem 3 illustrates one possible mission restriction. Problem 2 was rerun with a bound on the apogee thrust of 1440 m/sec, reflecting either a fixed apogee motor filling or a limit on tank capacity. This restriction increases the total cost by 9.5 m/sec, but it is still superior to the two-impulse maneuver of Problem 1.

In general, the required performance of the apogee motor, because of dispersions, is not realized exactly. Thus the preceding examples yield only lower bounds for the particular maneuvers described. Since even for faulty apogee motor performance the mission must succeed, it is important to find minimum fuel controls for given, that is, nonoptimal primary orbits as well.

Problem 4 displays the effect of errors both in magnitude and direction of the apogee motor thrust vector. Total thrust is taken to be 1430 m/sec (vs 1451 m/sec for Problem 2). Two corrective impulses ($n = 8$) in the primary and secondary orbits are assumed. As before, the rendezvous conditions, Eq. (23), must be satisfied. For this example the total fuel required is increased to 1535.5 m/sec.

Finally, Problem 5 again treats the case of given apogee thrust as in Problem 4, but with two bounds on the controls ($q = 2$). The time t_2 of the first correction must occur at least 8 hr after insertion into the primary orbit, for example, to allow time for orbit measurement, and the third corrective impulse c_3 is required to be less than 60 m/sec, defining a possible system limitation. Control bounds are often difficult to satisfy. This particular example illustrates the point. Not only is the total cost increased to 1544.8 m/sec, but convergence was also more difficult to obtain.

Table 3 contains initial and converged values of the control parameters together with the costs for Problems 1–5.

The results described in this section were obtained on a Telefunken TR440 computer using double precision arithmetic. Convergence of the algorithm was specified by the stopping conditions

$$\|L_x(\bar{x}, \lambda)\|_n \leqq \varepsilon_1 \qquad \|g(\bar{x})\|_p \leqq \varepsilon_2$$

Resetting of the matrix H in Eqs. (16) and (17) was enforced after every n projection updates. The "complementary DFP" rank two update[26] for H was employed together with a gradient linear search by cubic fit which required, on the average, four to five function evaluations, i.e., evaluation of $L(x, \lambda)$ for given x and λ.

Total computing time has been decreased by use of an idea due to Kelley and Speyer.[4] In a neighborhood of a constrained minimum, the change in the Lagrange multiplier is zero to first order. Therefore, λ may be held fixed during the linear search for β, avoiding repeated inversion of the matrix $g_x H g_x^*$.

Inequality constraints of the form

$$g_i(x) \leqq 0 \qquad i = p+1, \ldots, p+q$$

have been accommodated by building the gradients from the ε-active constraints defined by the index set

$$\{i \mid g_i(x) + \varepsilon \geqq 0 \qquad i = p+1, \ldots, p+q\}$$

where ε is a small positive constant. This device helps to avoid "jamming" and aids convergence of the algorithm.[27,28] The

Table 4 Convergence properties of the projection-restoration algorithm

Problem	Cycles[a]	Function evaluations Restoration	Projection	Computer time[b] (in seconds)	Parameters n	Constraints p	q
Earth-Mars	5	25	22	7	8	7	
1	6	25	19	6	8	6	
2	11	40	41	19	12	6	
3	15	61	66	31	12	6	1
4	15	79	100	27	8	6	
5	25	113	137	44	8	6	2

[a] Stopping tolerances: $\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = 10^{-6}$.

[b] Telefunken TR440 Computer (double precision).

remaining Kuhn-Tucker condition, that the Lagrange multiplier be non-negative, was not enforced during the descent. In addition, the search was terminated whenever a new constraint was violated.

Table 4 lists convergence information for the examples of this section. A cycle denotes the completion of both a restoration and a projection update. All examples used the values of the stopping tolerances shown. A limit of 50 was set on the maximum number of cycles.

For purposes of comparison, all of the preceding problems were also run with the Fletcher-Reeves implementation of conjugate gradient descent. For the Earth-Mars case, the performance was equivalent to that obtained with quasi-Newton descent. However, for none of the satellite positioning problems was convergence obtained using the same stopping tolerances. This fact points out a particular aspect of the satellite problem, namely that the minima are, in general, very flat, thus demanding high accuracy in the determination of the search directions. The entries for Problems 4 and 5 give an indication of this property. Although the dimension is still small ($n = 8$), a large number of function evaluations was required before convergence was obtained. In these cases, not only was the quasi-Newton method superior to the conjugate gradient method, but it was also necessary for obtaining a solution to the problem.

Conclusions

The projection-restoration technique is an effective method for solving finite dimensional nonlinear constrained optimization problems arising in applications. Combined with quasi-Newton descent, rapid convergence to constrained minima is obtained.

In applications to several problems of trajectory rendezvous, subject in addition to control bounds, the convergence properties of the algorithm were demonstrated. For the positioning problem, quasi-Newton descent was superior to conjugate gradient implementation. In recent years, constraints arising in applications have usually been handled by means of penalty functions involving, in general, difficulties in choosing the penalty constants and ill-conditioning of the Hessian matrix of the objective functional as the minimum is approached. The results presented in this paper show that direct treatment of constraints by projection-restoration constitutes a viable alternative.

Finally, the reliability of the projection-restoration algorithm can be further improved by a relatively simple modification involving the use of an augmented Lagrangian that combines linear and quadratic terms of the constraints in the functional to be minimized. An investigation of this "method of multipliers" and its application appears in the references.[29]

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